Section 5.3

Definition of Inverse Function: A function g is the **inverse function** of the function f when

$$f(g(x)) = x \text{ for each } x \text{ in the domain of } g$$

and

$$g(f(x)) = x \text{ for each } x \text{ in the domain of } f.$$

The function g is denoted by f^{-1} (read "f inverse").

Reflective Property of Inverse Functions: The graph of f contains the point (a, b) if and only if the graph of f^{-1} contains the point (b, a).

Theorem: The Existence of an Inverse Function

1. A function has in inverse if and only if it is one-to-one.
2. If f is strictly monotonic on its entire domain, then it is one-to-one and therefore has an inverse function.

Guidelines for Finding an Inverse Function

1. Use the Theorem above to determine whether the function $y = f(x)$ has an inverse function.
2. Solve for x as a function of y: $x = g(y) = f^{-1}(y)$.
3. Interchange x and y. The resulting equation is $y = f^{-1}(x)$.
4. Define the domain of f^{-1} as the range of f.
5. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

The Derivative of an Inverse Function: Let f be a function that is differentiable on an interval I. If f has an inverse function g, then g is differentiable at any x for which $f'(g(x)) \neq 0$. Moreover,

$$g'(x) = \frac{1}{f'(g(x))}, \quad f'(g(x)) \neq 0.$$

1) Show that the functions $f(x) = \sqrt{4x+5}$ and $g(x) = \frac{x^2-5}{4}$ are inverse functions of each other.

2) Which of the following functions has an inverse function?

a) $f(x) = 2x^3 + x - 1$

b) $f(x) = 3x^3 - x - 2$

3) Find the inverse function of

$$f(x) = \sqrt{5x + 7}$$

Identify the domain and range of $f(x)$ and $f^{-1}(x)$.

4) Show that the cosine function $f(x) = \cos x$ is not one-to-one on the entire real line. Then find the largest interval for which f is strictly monotonic (either increasing or decreasing).

5) Let $f(x) = x^3 + 2x - 1$.

a) What is the value of $f^{-1}(x)$ when $x = 2$?

b) What is the value of $(f^{-1})'(x)$ when $x = 2$?

Homework for 5.3: #5, 17, 23, 45, 51, 69, 73, 75